LOG-PLURICANONICAL SYSTEMS OF SMOOTH PROJECTIVE SURFACES (Workshop for young mathematicians on Several Complex Variables)

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LOG-PLURICANONICAL SYSTEMS OF SMOOTH PROJECTIVE SURFACES

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1 Introduction

Let X be a smooth projective variety and K_X be a canonical divisor of X. Then X is called of general type when pluricanonical system $|mK_X|$ defines a birational embedding of X for some positive integer m. The behavior of the pluricanonical systems is important to study varieties of general type. For example, there is such a problem :

Problem 1.1 *Let X be a smooth projective variety of general type. Find a positive ingeger m_0 such that for every $m \geq m_0$, $|mK_X|$ gives a birational map from X into a projective space.* □

In the case $\dim X = 1$, it is well known that $|3K_X|$ gives a projective embedding. In the case $\dim X = 2$, E. Bombieri proved that $|5K_X|$ gives a birational embedding([1]). Recently, H. Tsuji showed that there exists an integer $\nu_n$which depends only on $n = \dim X$ and satisfies above problem([6,7]). But when $\dim X \geq 4$, effective value of ν_n is unknown. Even if $\dim X = 3$, the value of ν_n becomes an astronomical number, and it is supposed that the value computed in [7] is not best-possible.

I am interested in this problem for open surfaces. As long as I know, such a situation is not studied yet.

Definition 1.1 *Let X be a surface and D be a divisor with normal clossings. Then the pair (X, D) is called log-surface. If the liniar system $|K_X + D|$ is big, we say that (X, D) is of log-general type.* □

Now we state the problem more precisely.

Problem 1.2 *Let (X, D) be a smooth projective surface of log-general type. Find a positive ingeger m_0 such that for every $m \geq m_0$, $|m(K_X + D)|$ gives a birational map from X into a projective space.* □

The main purpose of this paper is to answer the weeker version of this problem. Our result shows the value of m_0 for a given surface. But it depends on X and divisor D.

Theorem 1.1 *Let (X, D) be a smooth projective surface of log-general type, and let $K_X + D = P + E$ be a Zariski-decomposition of $K_X + D$, where P is a nef part and E is effective part of the decomposition. Then $|m(K_X + D)|$ defines a birational map from X into projective space unless*

$$m \geq \frac{6\sqrt{2}}{\sqrt{P^2}} + 4 \ .$$

□

Remark 1.1 *In the case (X, D) is log-general type, $P^2 > 0$ holds* ([3]).

2 Terminology

In this section we introduce a singular harmitian metric and some results we use after. See [2] for more details.

Definition 2.1 *Let L be a holomorphic line bundle on X, h_0 be a C^∞-hermitian metric on L, and φ be a L^1_{loc}-function on X. Then we call $h = e^{-\varphi} \cdot h_0$ a singular hermitian metric wiht respect to φ. φ is called a weight function of h.* □

Definition 2.2 *We define a curvature current $i\Theta_h$ of a singular hermitian line bundle (L, h) as follows:*

$$i\Theta_h := i\partial\bar{\partial}\varphi \ + \ i\Theta_{h_0}$$

where $\partial\bar{\partial}$ is taken as a distribution and $i\Theta_{h_0}$ is the curvature form of (L, h_0) in usual sense.

A singular hermitian line bundle is said to be positive , if the curvature current $i\Theta_h$ becomes a measure which takes values in semipositive-defined hermitian matrix. □

Next we introduce a concept of multiprier ideal sheaves. Let $U \subset X$ be an open set and $\mathcal{O}(U)$ be the set of holomorphic functions on U. Then

$$I_U(h) := \left\{ f \in \mathcal{O}(U) \;\middle|\; \int_U e^{-\varphi}\, |f|^2\, dV \;<\; +\infty \right\}$$

becomes a presheaf when U runs all open subsets of X. We put $I(h)$ as the sheafication of $I_U(h)$. $I(h)$ is called the multiplier ideal sheaf with respect to h. The following theorem which is a variant of Kodaira 's vanishing theorem is due to A.Nadel([5]).

Theorem 2.1 *Let (X,ω) be a Kähler manifold and (L,h) be a singular hermitian line bundle on X. Assume that $i\Theta_h \geq \varepsilon_0\, \omega$ for some $\varepsilon_0 > 0$. Then*

$$H^q(X, \mathcal{O}_X(K_X + L) \otimes I(h)) = 0 \qquad (q \geq 1) .$$

□

3 Proof of Theorem1.1

In this section, we show the outline of the proof of Theorem1.1. The proof is made along [6, section2], so please refer to [6] for detail.

Let (X, D) be a smooth projective surface of log-general type and $x, y \in X$, $x \neq y$ be generic two points. Assume that there exists a singular hermitian metric $h_{x,y}$ on $m(K_X + D) + D$ such that:

1. $x, y \in \mathrm{supp}\mathcal{O}_X/I(h_{x,y})$
2. One of the x or y, say x, is an isolated point of supp $\mathcal{O}_X/I(h_{x,y})$
3. $i\Theta_h \geq \varepsilon_0\, \omega$ for some $\varepsilon_0 > 0$.

We consider the long exact sequence:

$$\cdots \longrightarrow H^0(X, \mathcal{O}_X((m+1)(K_X + D)))$$

$$\longrightarrow H^0(X, \mathcal{O}_X((m+1)(K_X + D)) \otimes \mathcal{O}/I(h_{x,y}))$$

$$\longrightarrow H^1(X, \mathcal{O}_X(K_X + m(K_X + D) + D) \otimes I(h_{x,y})) \longrightarrow \cdots \quad .$$

Where $H^1(X, \mathcal{O}_X(K_X+m(K_X+D)+D)\otimes I(h_{x,y})) = 0$ by Nadel 's vanishing theorem, hence we get surjection and we can conclude that there exists some $\sigma \in H^0(X, \mathcal{O}_X((m+1)(K_X + D)))$ such that $\sigma(y) = 0$ and $\sigma(x) \neq 0$. This shows that $\Phi_{|(m+1)(K_X+D)|}$ separates x and y. Therefore to prove theorem1.1, we have only to compute the value m such that we can construct a singular hermitian metric $h_{x,y}$ on $(m+1)(K_X + D)$ which satisfies the condition 1, 2 and 3 above for arbitrary distinct two points $x, y \in U$, for some nonempty Zariski open subset $U \subset X$.

3.1 Construction of $h_{x,y}$

Let $K_X + D = P + E$ be a Zariski-decomposition of $K_X + D$, where P is the nef part and E is the effective part of the decomposition. We put X° as follows:

$$X^\circ := \{\, p \in X \mid p \notin Bs|mP| \text{ and for some m, } |mP| \text{ gives biholomorpic near } p\}$$

Then X° is a nonempty Zariski open set of X.

We take arbitrary $x, y \in X^\circ$ and we set $\mathcal{M}_{x,y} := \mathcal{M}_x \otimes \mathcal{M}_y$, where $\mathcal{M}_x$ and $\mathcal{M}_y$ are the maximum ideal sheaf of the points x, y respectively. By considering a cohomology exact seqence and comparing the dimension of $H^0(X, \mathcal{O}_X(mP))$ and $H^0(X, \mathcal{O}_X(mP) \otimes \mathcal{O}_X/\mathcal{M}_{x,y}^{\otimes\lceil\sqrt{\frac{P^2}{2}}\cdot(1-\varepsilon)m\rceil})$, we can show the following :

Proposition 3.1 *For arbitrary small $\varepsilon > 0$,*

$$\dim H^0(X, \mathcal{O}_X(mP) \otimes \mathcal{O}_X/\mathcal{M}_{x,y}^{\otimes\lceil\sqrt{\frac{P^2}{2}}\cdot(1-\varepsilon)m\rceil}) \geq 1$$

holds if we take m sufficiently large. □

We take $\sigma_0 \in H^0(X, \mathcal{O}_X(m_0P) \otimes \mathcal{O}_X/\mathcal{M}_{x,y}^{\otimes\lceil\sqrt{\frac{P^2}{2}}\cdot(1-\varepsilon_0)m_0\rceil})$ for sufficiently small ε_0 and sufficiently large m_0.

If we set $h_0 := \frac{1}{|\sigma_0|^{2/m_0}}$, then h_0 is a singular hermitian metric on P with positive curvature.

We set α_0 as follows :

$$\alpha_0 := \inf\{\, \alpha > 0 \mid \operatorname{supp} \mathcal{O}_X/I(h_0^\alpha) \ni x, y \,\} .$$

σ_0 has zeros of order at least $\lceil\sqrt{\frac{P^2}{2}}\cdot(1-\varepsilon)m\rceil$, so we get $\alpha_0 \leq \sqrt{\frac{2}{P^2}} \cdot \frac{2}{1-\varepsilon_0}$.

Next we decrease α_0 a little bit. Then one of the following two cases occurs.

Case 1. supp $\mathcal{O}_X/I(h_0^{\alpha-\delta_0})$ does not include either x nor y.

Case 2. supp $\mathcal{O}_X/I(h_0^{\alpha-\delta_0})$ includes one of x or y, say x.

In Case 1, we can consider a minimal center of log canonical singularities at x. Let X_1 be a minimal center at x. In this case one of following two cases occurs.

Case 1-1. *supp* $\mathcal{O}_X/I(h_0^{\alpha-\delta_0})$ does not include either x nor y.

Case 1-2. Otherwise.

We shall explain Case 1-1. (Other cases are easier to prove.)

Note that $(X_1 \cdot P) > 0$ because X_1 passes through $x \in X^\circ$.

Proposition 3.2 *For arbitrary small* $\varepsilon > 0$,

$$\dim H^0(X_1, \mathcal{O}_{X_1}(mP) \otimes \mathcal{O}_X/\mathcal{M}_{x,y}^{\otimes\lceil \frac{(X_1\cdot P)}{2}\cdot(1-\varepsilon)m\rceil}) \geq 1$$

holds if we take m *sufficiently large.* □

The proof of Proposition3.2 is the same as the proof of Proposition3.1.

We take $\tilde{\sigma_1} \in H^0(X, \mathcal{O}_{X_1}(m_1P) \otimes \mathcal{O}_{X_1}/\mathcal{M}_{x,y}^{\otimes\lceil \frac{(X_1\cdot P)}{2}\cdot(1-\varepsilon_1)m_1\rceil})$ for sufficiently small ε_1 and sufficiently large m_1.

Because P is nef big, P has a decomposition $P = \mathcal{A}+\mathcal{E}$ by Kodaira 's lemma. Where $\mathcal{A}$ is a $\mathbf{Q}$-ample divisor and $\mathcal{E}$ is a $\mathbf{Q}$-effective divisor. We take integer l_1 sufficiently large so that $L_1 := l_1\cdot\mathcal{A}$ is $\mathbf{Z}$-very ample. Let $\tau \in H^0(X_1, \mathcal{O}_{X_1}(L_1))$ be a section which is not zero section. then

$$\tilde{\sigma_1} \otimes \tau \in H^0(X_1, \mathcal{O}_{X_1}(mP+L_1) \otimes \mathcal{O}_X/\mathcal{M}_{x,y}^{\otimes\lceil \frac{(X_1\cdot P)}{2}\cdot(1-\varepsilon)m\rceil})$$

holds.

Proposition 3.3 *For* $m \geq 0$,

$$H^0(X, \mathcal{O}_X(mP+L_1)) \longrightarrow H^0(X_1, \mathcal{O}_{X_1}(mP+L_1))$$

is surjective if we take l_1 *sufficiently large.* □

Proof. Set $\varphi = \alpha_0 \log \frac{h_0}{h_P}$. Where h_P is arbitrary C^∞-hermitian metric on P. We consider $\varphi \cdot h_{L_1} \cdot h_{K_X^{-1}}$. This is a singular hermitian metric on $L_1 - K_X$. If we take l_1 sufficiently large, the curvature is strictly positive and $\mathcal{O}_X/I(\varphi) = \mathcal{O}_{X_1}$. Since P is nef, we get $H^1(X, \mathcal{O}_X(mP+L_1) \otimes I(h_{mP+L_1-K_X})) = 0$. This completes the proof. ■

By using this proposition, we extend $\tilde{\sigma_1} \otimes \tau$ to

$$\sigma_1 \in H^0(X, \mathcal{O}_X((m_1+l_1)P)) .$$

Let $\{\rho_j\}$ be generator of $\mathcal{O}_X((m_1+l_1)\cdot\mathcal{A}) \otimes \mathcal{I}_X$. We put

$$h_1 := \frac{1}{(|\sigma_1|^2 + \sum|\rho_j|^2)^{1/(m_1+l_1)}} .$$

We take m_1 sufficiently large so that $\frac{l_1}{m_1} \leq \delta_0 \frac{(X_1\cdot P)}{2}$ holds.

Proposition 3.4 *Let* $\alpha_1 = \inf\{\alpha > 0 \mid \operatorname{supp}\mathcal{O}_X/I(h_0^{\alpha_0-\delta_0}\cdot h_1^{\alpha}) \ni x, y\}$. *Assume* x *and* y *be regular points of* X_1. *Then*

$$\alpha_1 \leq \frac{2}{(X_1\cdot P)} + O(\delta_0)$$

Proof. We can choose a neighborhood U of x and a local coodinate system (z_1, z_2) on U such that

$$U \cap X_1 \;=\; \{\, p \in U \,|\, z_1(p) = 0 \,\} \;=\; \{\, (0, z_2) \,\}$$

holds.Then we get

$$||\sigma_1||^2 + \sum ||\rho_j||^2 \;\le\; C \cdot \left(|z_1|^2 \;+\; |z_2|^{2 \cdot \lceil \frac{(X_1,P)}{2} \cdot (1-\varepsilon_1) \cdot m_1 \rceil} \right) \;,$$

here $|| \cdot ||$ is taken with respect to some C^∞-hermitian metric on $(m_1 + l_1)P$, and C is a constant depending on the norm $|| \cdot ||$. By the construction of σ_0,

$$||\sigma_0||^{\frac{2}{m_0} \cdot (\alpha_0 - \delta_0)} \;\le\; O(|z_1|^{2-\delta_0})$$

also holds on some neighborhood of generic points of $U \cap X_1$. Hence we get

$$\alpha_1 \;\le\; \frac{(m_1 + l_1)}{m_1} \cdot \frac{2}{(X_1 \cdot P)} \;+\; O(\delta_0) \;.$$

From the assumption $\frac{l_1}{m_1} \le \delta_0 \, \frac{(X_1 \cdot P)}{2}$, we conclude the statement of the proposition. ■

Remark 3.1 *Even if x and y are not regular points of X_1, we can show above result is true by taking $\dot{x}$ and $\dot{y}$ as regular points of X_1 and letting $\dot{x} \longrightarrow x$ and $\dot{y} \longrightarrow y$.*

Lemma 3.1 *$|\, m\,(K_X + D)\,|$ separates x and y for $m \ge \lceil \alpha_0 \,+\, \alpha_1 \rceil \,+ 1$.* □

Proof. By the eqation

$$m\,(K_X + D) \;=\; K_X + (m-1)\,P + (m-1)\,E + D$$

and

$$(m-1)\,P \;=\; \{\, (\alpha_0 - \delta_0) + \alpha_1 \,\}\,P \;+\; \{\, m - 1 - \,(\alpha_0 - \delta_0 + \alpha_1)\,\}\,(\,\mathcal{A} \,+\, \mathcal{E}\,) \;\;,$$

we can equip a singular hermitian metric $h_{x,y}$ by

$$h_{x,y} \;:=\; h_0^{\alpha_0 - \delta_0} \,\cdot\, h_1^{\alpha_1} \,\cdot\, h_{\mathcal{A}}^{m-1-\,(\alpha_0 - \delta_0 + \alpha_1)} \,\cdot\, h_{\mathrm{eff}} \;\;,$$

where $h_{\mathcal{A}}$ is a C^∞-hermitian metric of $\mathbf{Q}$-ample divisor $\mathcal{A}$ and h_{eff} is a semipositive singular hermitian metric which comes from the other components. Then by the construction of h_0 and h_1, $h_{x,y}$ satisfies the following conditions :

1. $x, y \in \mathrm{supp}\mathcal{O}_X/I(h_{x,y})$
2. One of the x or y, say x, is an isolated point of supp $\mathcal{O}_X/I(h_{x,y})$.
3. $i\Theta_h \geq \varepsilon_0\,\omega$ for some $\varepsilon_0 > 0$.

So there exists some $\sigma \in H^0\left(X\,,\,m\,(K_X+D)\right)$ such that $\sigma(y) = 0$ and $\sigma(x) \neq 0$, or $\sigma(x) = 0$ and $\sigma(y) \neq 0$. This completes the proof. ■

Cororally 3.1 $|\,m\,(K_X + D)\,|$ *separates* x *and* y *for*

$$m \geq \frac{2\sqrt{2}}{\sqrt{P^2}} + \frac{2}{(X_1 \cdot P)} + 1\ .$$

□

3.2 Construction of X_1 as a family

Our construction of X_1 is depending on the choice of the points x and y. Therefore it seems that the value of (X_1, P) is also depending on x and y. But in fact, (X_1, P) is independent of generic choice of $x, y \in X$. We explain it in this subsection.

Let $\Delta_X \subset X \times X$ be a diagonal set. We set $B \subset X \times X$ and $Z \subset B \times X$as follows:

$$B\ :=\ X^\circ \times X^\circ - \Delta_X$$

$$Z\ :=\ \left\{\ (z_1, z_2, z_3)\ \middle|\ x_3\ =\ x_1\ \text{ or }\ x_2\ =\ x_1\ \right\}\ .$$

Let $p\ :\ X \times B\ \longrightarrow X$ and $q\ :\ X \times B\ \longrightarrow B$ be the frist and second projection respectively. We consider

$$q_*\left(\mathcal{O}_{X\times B}(m_0\,p^*P) \otimes \mathcal{I}_Z^{\otimes\lceil\sqrt{\frac{P^2}{2}}\cdot(1-\varepsilon)\,m\rceil}\right)$$

instead of

$$H^0\left(X\,,\,\mathcal{O}_X(mP)\ \otimes \mathcal{O}_X/\mathcal{M}_{x,y}^{\otimes\lceil\sqrt{\frac{P^2}{2}}\cdot(1-\varepsilon)\,m\rceil}\ \right),$$

where $\mathcal{I}_Z$ denotes the ideal sheaf of Z. For a sufficiently large integer m_0 and sufficiently small ε, we take $\tilde{\sigma}_0$ as a nonzero global meromorphic section of

$$q_*\left(\mathcal{O}_{X\times B}(m_0\,p^*P) \otimes \mathcal{I}_Z^{\otimes\lceil\sqrt{\frac{P^2}{2}}\cdot(1-\varepsilon)\,m\rceil}\right)\ .$$

$$\tilde{h}_0 := \frac{1}{|\tilde{\sigma}_0|^{2/m_0}} ,$$

then h_0 is a singular hermitian metric on P (but curvature current of $\tilde{h}_0$ may not be positive). We shall replace α_0 by

$$\tilde{\alpha}_0 \; = \; \inf\{ \, \alpha > 0 \, | \text{ The generic points of } Z \subset \operatorname{Spec}(\mathcal{O}_{X\times B}/\mathcal{I}(\tilde{h}_0^{\alpha})) \; \} \; .$$

Then for every small $\delta > 0$, there exists a Zariski open subset U of B such that $\tilde{h}_0 \,|_{X\times\{b\}}$ is well-defined for every $b \in U$, and

$$b \not\subseteq \operatorname{Spec}(\mathcal{O}_{X\times\{b\}}/\mathcal{I}(\tilde{h}_0^{\tilde{\alpha}_0-\delta})) \; ,$$

where we have identified b with distinct two points in X. By the construction of α_0, we can see

$$b \subsetneq \operatorname{Spec}(\mathcal{O}_{X\times\{b\}}/\mathcal{I}(\tilde{h}_0^{\tilde{\alpha}_0}))$$

for every $b \in B$. Let $\tilde{X}_1$ be a minimal center of logcanonical singularities of $(X \times B, \frac{\tilde{\alpha}_0}{m_0}(\tilde{\sigma}_0)\,)$ at the generic point of Z (although $(\tilde{\sigma}_0)$ may not be effective, but this is still meaningfull in this case because of our construction of $\tilde{\sigma}_0$). Then $\tilde{X}_1 \cap q^{-1}(b)$ is almost a minimal center at $b := \{\text{Distinct two points in } X^\circ\}$ which we construct in the last subsection. Remark that $\tilde{X}_1 \cap q^{-1}(b)$ may not be irreducible even for a general $b \in B$. But if we take a suitable finite cover

$$\phi_0 \; : \; B_0 \longrightarrow B \;\; ,$$

on the base change $X \times_B B_0$, $\hat{X}_1$ defines a family of irreducible subvarieties

$$f \; : \; \hat{X}_1 \; \longrightarrow \; U_0$$

of X parametrized by a nonempty Zariski open subset U_0 of $\phi_0^{-1}(U)$.

From above arguments, we see that $\{X_1\}$'s are numerically equivalent to each other when we move $b = (x, y) \in X^\circ \times X^\circ - \Delta_X$ generically. The intersection number (X_1, P) takes value in Q, therefore (X_1, P) is constant if we choose $b = (x, y)$ generically. Hence we get:

Proposition 3.5 *$|m(K_X + D)|$ defines a birational map from X to a projective space if*

$$m \geq \frac{2\sqrt{2}}{\sqrt{P^2}} + \frac{2}{(X_1 \cdot P)} + 1 \; .$$

3.3 An estimate of $(X_1 \cdot P)$

To complete the proof of Theorem1.1 , we have to estimate $(X_1 \cdot P)$.

We consider the self-intersection number $(X_1)^2$. Then there are three possibilities:

Case 1. $(X_1)^2 > 0$

Case 2. $(X_1)^2 = 0$

Case 3. $(X_1)^2 < 0$

Let (x, y) and $(\grave{x}, \grave{y})$ be pair of distinct two points of X°. We put X_1 and $\grave{X}_1$ as a minimal center at (x, y) and $(\grave{x}, \grave{y})$ respectively. If we take (x, y) and $(\grave{x}, \grave{y})$ general, X_1 and $\grave{X}_1$ have no common irreducible components. Since X_1 and $\grave{X}_1$ are numerically equivalent, we get

$$(X_1)^2 = (\grave{X}_1)^2 = (X_1, \grave{X}_1) \geq 0 .$$

So we have only to consider the case $(X_1)^2 \geq 0$.

i) In the case $(X_1)^2 > 0$.

By the Hodge index theorem, we get

$$(X_1, P) \geq \sqrt{(X_1)^2} \cdot \sqrt{(P)^2} .$$

Since X_1 is an integral divisor, $(X_1)^2$ takes balue in $\mathbf{Z}$. As a consequence we have $(X_1)^2 \geq 1$ and

$$\frac{1}{(X_1, P)} \leq \frac{1}{\sqrt{P^2}} .$$

So in this case the proof of Theorem1.1 is completed.

ii) In the case $(X_1)^2 = 0$.

Let $\mathcal{N}_{X_1}$ be a normal bundle of X_1. Then we have $\mathcal{N}_{X_1} = -X_1|_{X_1}$ and $\deg_{X_1} \mathcal{N}_{X_1} = -(X_1)^2 = 0$. So we see that the normal bundle of X_1 is trivial. Furthermore, X_1 can move. As a consequence, we can conclude existence of a fibration of X:

$$\pi : X \longrightarrow S ,$$

where S denotes some algebraic curve. By the definition of α_0, $\alpha_0 P - \pi^*(p_x) - \pi^*(p_y)$ is a pseudeffective line bundle on X. Here p_x and p_y denote the point $\pi(x)$ and $\pi(y)$ respectively. Because $\deg_S K_S = 2g_S - 2 \geq -2$, we have

$$\alpha_0 P \geq \pi^*(2 \text{ points in } S) \geq -\pi^* K_S$$

and

$$H^0(X, \mathcal{O}_X(m(1+\alpha_0)(K_X+D))) \supset H^0(X, \mathcal{O}_X(m(K_X+D-\pi^*K_S))) \ .$$

Recall that we regard $H^0(X, \mathcal{O}_X(m(K_X+D-\pi^*K_S)))$ as a subset of $H^0(X, \mathcal{O}_X(m(1+\alpha_0)(K_X+D)))$ by using natural injective map derived from the sheaf exact sequence

$$0 \longrightarrow \mathcal{O}_X(m(K_X+D-\pi^*K_S)) \longrightarrow \mathcal{O}_X(m(1+\alpha_0)(K_X+D)) \ ,$$

and hereafter we will often use such notation. By the definition of Zariski decomposition and above inclusion, we have the natural injection

$$\phi : H^0(X, \mathcal{O}_X(m(K_X+D-\pi^*K_S))) \longmapsto H^0(X, \mathcal{O}_X(m(1+\alpha_0)P)) \ ,$$

if we let m be an integer such that $m(1+\alpha_0)P$ is a $\mathbf{Z}$-divisor.

The divisor $\pi_*(K_X+D-\pi^*K_S)$ is semipositive by Kawamata 's semipositivity theorem[4 , theorem 1], hence we get

$$H^0(S, \mathcal{O}_S(m\pi_*(K_X+D-\pi^*K_S))) \longrightarrow m\pi_*(K_X+D-\pi^*K_S) \otimes \mathcal{O}_S/\mathbf{m}_p$$

is surjective for sufficiently large m. From the above surjection, we get

$$H^0(X, \mathcal{O}_X(m(K_X+D-\pi^*K_S))) \longrightarrow H^0(\pi^{-1}(p), \mathcal{O}_{\pi^{-1}(p)}(m(K_X+D-\pi^*K_S)|_{\pi^{-1}(p)}))$$

is also surjective. Since $\pi^*K_S|_{\pi^{-1}(p)}$ is trivial bundle, we have a surjection :

$$H^0(X, \mathcal{O}_X(m(K_X+D-\pi^*K_S))) \longrightarrow H^0(\pi^{-1}(p), \mathcal{O}_{\pi^{-1}(p)}(m(K_X+D)|_{\pi^{-1}(p)})) \ .$$

Let us consider $H^0(X, \mathcal{O}_X(m(1+\alpha_0)P))\Big|_{\pi^{-1}(p)}$. By the natural injective map ϕ, we can see

$$H^0(X, \mathcal{O}_X(m(1+\alpha_0)P))\Big|_{\pi^{-1}(p)} \supset H^0(\pi^{-1}(p), \mathcal{O}_{\pi^{-1}(p)}(m(K_X+D)|_{\pi^{-1}(p)}))$$

holds.

Let σ_1 and σ_2 be a global section of $H^0(X, \mathcal{O}_X(m(K_X+D-\pi^*K_S)))$ such that $\sigma_1|_{\pi^{-1}(p)}$ and $\sigma_2|_{\pi^{-1}(p)}$ are linearly independent. Then, if we take a general fiber $\pi^{-1}(p)$, $\phi(\sigma_1)|_{\pi^{-1}(p)}$ and $\phi(\sigma_2)|_{\pi^{-1}(p)}$ are also linearly independent.

Hence we get an inequality on dimensions of holomorphic sections:

$$\dim H^0(\,X,\,\mathcal{O}_X(\,m(1+\alpha_0)P\,)\,)\,\Big|_{\pi^{-1}(p)}$$

$$\geq\ \dim H^0(\,\pi^{-1}(p),\,\mathcal{O}_{\pi^{-1}(p)}(\,m\pi_*(K_X+D)|_{\pi^{-1}(p)}\,)\,)\ .$$

We know the asymptotic relations :

$$\dim H^0(\,X,\,\mathcal{O}_X(\,m(1+\alpha_0)P\,)\,)\,\Big|_{\pi^{-1}(p)}\ \sim\ m(1+\alpha_0)(P,\pi^{-1}(p))$$

and

$$\dim H^0(\,\pi^{-1}(p),\,\mathcal{O}_{\pi^{-1}(p)}(\,m\pi_*(K_X+D)|_{\pi^{-1}(p)}\,)\,)\ \sim\ m(K_X+D,\pi^{-1}(p))\ ,$$

when we keep $m(1+\alpha_0)P$ be integral divisor and letting m to be sufficiently large. Letting $m\longrightarrow\infty$, we see

$$(1+\alpha_0)(\pi^{-1}(p),P)\ \geq\ (\pi^{-1}(p),K_X+D)\ .$$

By definition, $(\pi^{-1}(p),P)\ =\ (X_1,P)$ and $(\pi^{-1}(p),K_X+D)\ =\ (X_1,K_X+D)$ holds. Hence we have

$$(1+\alpha_0)(X_1,P)\ \geq\ (X_1,K_X+D)\ .$$

If we take a general fiber, $(K_X+D)\,|_{X_1}$ becomes a big divisor and

$$\deg_{X_1}(K_X+D)\ =\ (X_1,K_X+D)\ \geq 1$$

holds. Then we get an estimate for (X_1,P) :

$$1+\alpha_0\ \geq\ \frac{1}{(X_1,P)}\ .$$

Since $\alpha_0\ \leq\ \sqrt{\frac{2}{P^2}}\cdot\frac{2}{1-\varepsilon_0}$, then we have

$$\frac{2\sqrt{2}}{\sqrt{P^2}}+\frac{2}{(X_1\cdot P)}\ +\ 1\ \leq\ \frac{2\sqrt{2}}{\sqrt{P^2}}\ +\ \frac{\sqrt{2}}{\sqrt{P^2}}\cdot\frac{4}{1-\varepsilon_0}\ +\ 3\ ,$$

and this completes the proof of Theorem1.1.

References

[1] E. Bombieri, *Canonical models of surfaces of general type*, Pbul. I.H.E.S., **42** (1972), 171-219.

[2] J.-P. Demailly, *L^2 vanishing theorems for positive line bundles and adjunction theory*, Lecture Notes in Math., **1646**, Springer, (1996), 1-97.

[3] Y. Kawamata, *On the Classification of Non-complete Algebraic Surfaces*, Lecture Notes inMathematics, **732**, Springer-Verlag, (1979), 215-232.

[4] Y. Kawamata, *Kodaira Dimension of Algebraic Fiber Spaces Over Curves*, Invent. math., **66**, (1982), 57-71.

[5] A.M. Nadel, *Multiplier ideal sheaves and existence of Kähler-Einstein metrics of positive scalar curvature*, Ann. of Math. **132** (1990), 549-596.

[6] H. Tsuji, *Pluricanonical systems of projective varieties of general type*, math.AG/9909021.

[7] H. Tsuji, *Pluricanonical systems of projective 3-folds of general type*, math.AG/0204096.

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